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1988 J. Phys. A: Math. Gen. 21 L1117

(http://iopscience.iop.org/0305-4470/21/23/002)

J. Phys. A: Math. Gen. 21 (1988) L1117-L1121. Printed in the UK

LETTER TO THE EDITOR

Non-local ansätze for the Dirac equation

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Received 5 July 1988

Abstract. Using non-local (non-Lie) symmetry of the linear Dirac equation we have constructed a number of new ansätze reducing it to systems of ordinary differential equations.

It is well known (see e.g. [1]) that the Poincaré group P(1, 3) is a maximal local (in Lie's sense) invariance group of the linear Dirac equation

$$(\mathrm{i}\gamma_\mu\partial_\mu + m)\psi(x) = 0 \qquad m = \text{constant} \tag{1}$$

where $\psi = \psi(x_0, \boldsymbol{x})$ is a four-component spinor, $\partial_\mu \equiv \partial/\partial x_\mu$, $\mu = \overline{0,3}$ and γ_μ are imaginary 4×4 matrices satisfying the Clifford algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}I \equiv 2I\begin{cases} 1 & \mu=\nu=0 \\ -1 & \mu=\nu=\overline{1,3} \\ 0 & \mu\neq\nu. \end{cases}$$

In [2, 3] ansätze reducing the Dirac equation to systems of ordinary differential equations (ODE) were constructed, the subgroup structure of the group P(1, 3) investigated in detail by Patera *et al* [4, 5] being used.

As shown in [1, 6, 7] equation (1) possesses non-local (non-Lie) symmetry. So far this additional non-local symmetry has not been used to construct ansätze reducing the Dirac equation to systems of ODE. In the present paper we construct a number of such ansätze following an approach suggested in [3, 8].

If one puts

$$\Gamma_\mu = \mathrm{diag}(-\mathrm{i}\gamma_\mu, -\mathrm{i}\gamma_\mu) \qquad \Psi^{\mathrm{T}} = (\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)^{\mathrm{T}}$$

then equation (1) becomes

$$(\Gamma_\mu\partial_\mu - m)\Psi(x) = 0. \tag{2}$$

It is common knowledge that the complete set of first-order symmetry operators of the Dirac equation (2) is not a Lie algebra. We have succeeded in picking out the subset which forms the Lie algebra of the Poincaré group:

$$P_\mu = [1 + \varepsilon(\Gamma_4 + \Gamma_5)]\partial^\mu + \varepsilon m(\Gamma_4 + \Gamma_5)\Gamma_\mu \tag{3}$$

$$J_{\mu\nu} = -x_\mu\partial^\nu + x_\nu\partial^\mu - \tfrac{1}{4}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu) \tag{4}$$

where ε = constant, $\partial^\mu = g^{\mu\nu}\partial_\nu$ for μ, $\nu = \overline{0,3}$ and

$$\Gamma_4 + \Gamma_5 = 2\begin{pmatrix} 0 & 0 \\ \gamma_0\gamma_1\gamma_2\gamma_3 & 0 \end{pmatrix}.$$

It is important to note that operators (3) generate a non-local group of transformations

$$\Psi' = [1 - \varepsilon m(\Gamma_4 + \Gamma_5)\theta^{\mu}\Gamma_{\mu}]\Psi + \varepsilon(\Gamma_4 + \Gamma_5)\theta_{\mu}\Psi_{x_\mu} \qquad x'_{\mu} = x_{\mu} + \theta_{\mu} \tag{5}$$

where θ_μ are group parameters.

According to [4, 8] there exists a correspondence between three-dimensional subalgebras of the algebra (3) and (4) and ansätze reducing the Dirac equation (2) to ODE. Omitting very cumbersome intermediate calculations we write the final result for the non-local ansätze for the spinor field.

1. $\langle P_0 + P_3, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2 + \tfrac{1}{2}\eta\Gamma_{03})]\varphi(\xi)$$

2. $\langle P_1, P_2, P_3 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3)]\varphi(x_0)$$

3. $\langle P_0, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2 - \Gamma_0 x_0)]\varphi(x_3)$$

4. $\langle J_{03}, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)] \exp(-\tfrac{1}{2}\Gamma_0\Gamma_3 \ln \xi)\varphi(x_0^2 - x_3^2)$$

5. $\langle J_{03}, P_0 + P_3, P_1 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \tfrac{1}{2}\eta\Gamma_{03})] \exp(-\tfrac{1}{2}\Gamma_0\Gamma_3 \ln \xi)\varphi(x_2)$$

6. $\langle J_{03} + \alpha P_2, P_0, P_3 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_3 x_3 - \Gamma_0 x_0)] \exp\{[\varepsilon\Gamma_{45}\Gamma_2 + (1/2\alpha)\Gamma_0\Gamma_3(\varepsilon\Gamma_{45} - 1)]x_2\}\varphi(x_1)$$

7. $\langle J_{03} + \alpha P_2, P_0 + P_3, P_1 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \tfrac{1}{2}\eta\Gamma_{03})] \exp\{x_2[m\xi^2\Gamma_2(1 + 2\varepsilon\Gamma_{45}) - \tfrac{1}{2}\xi\Gamma_2\Gamma_{03} - 3\alpha\varepsilon m\xi\Gamma_{03}\Gamma_{45} - \varepsilon m\xi^2\Gamma_2\Gamma_{45}\Gamma_0\Gamma_3]\}\varphi(\xi)$$

8. $\langle J_{12}, P_0, P_3 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_3 x_3 - \Gamma_0 x_0)] \exp[\tfrac{1}{2}\Gamma_1\Gamma_2 \tan^{-1}(x_1/x_2)]\varphi(x_1^2 + x_2^2)$$

9. $\langle J_{12} + \alpha P_0, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)] \exp\{[(1/2\alpha)\Gamma_1\Gamma_2(1 - \varepsilon\Gamma_{45}) - \varepsilon m\Gamma_{45}\Gamma_0]x_0\}\varphi(x_3)$$

10. $\langle J_{12} + \alpha P_3, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)] \exp\{[\varepsilon m\Gamma_{45}\Gamma_3 + (1/2\alpha)(1 - \varepsilon\Gamma_{45})\Gamma_1\Gamma_2]x_2\}\varphi(x_0)$$

11. $\langle J_{12} + P_0 + P_3, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)] \exp\{\tfrac{1}{2}\eta[\varepsilon m\Gamma_{45}\Gamma_{03} + \tfrac{1}{2}(1 - \varepsilon\Gamma_{45})\Gamma_1\Gamma_2]\}\varphi(\xi)$$

12. $\langle G_1, P_0 + P_3, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m\Gamma_{45}(\Gamma_2 x_2 + \tfrac{1}{2}\eta\Gamma_{03})] \exp[-(x_2/2\xi)\Gamma_{03}(\Gamma_1 + \varepsilon m x_2\Gamma_{45})]\varphi(\xi)$$

13. $\langle G_1, P_0 + P_3, P_1 + \alpha P_2 \rangle$

$$\Psi(x) = \exp\{\varepsilon m\Gamma_{45}[x_1(\Gamma_1 + \alpha\Gamma_2) + \tfrac{1}{2}\eta\Gamma_{03}]\} \times \exp\{[(\alpha x_1 - x_2)/\alpha\xi][\tfrac{1}{2}\Gamma_1\Gamma_{03} - \varepsilon m\xi\Gamma_{45}(\Gamma_1 + \alpha\Gamma_2)]\}\varphi(\xi)$$

14. $\langle G_1+P_2,\, P_0+P_3,\, P_1\rangle$

$$\Psi(x)=\exp[\varepsilon m\Gamma_{45}(x_1\Gamma_1+\tfrac{1}{2}\eta\Gamma_{03})]$$
$$\times\exp\{x_2[\varepsilon m\Gamma_{45}(\Gamma_2-\xi\Gamma_1)+\tfrac{1}{2}(\varepsilon\Gamma_{45}-1)\Gamma_{03}\Gamma_1]\}\varphi(\xi)$$

15. $\langle G_1+P_0,\, P_0+P_3,\, P_2\rangle$

$$\Psi(x)=\exp[\varepsilon m\Gamma_{45}(x_2\Gamma_2+\tfrac{1}{2}\eta\Gamma_{03})]$$
$$\times\exp[mx_1(\Gamma_1+\xi\Gamma_{03}+3\varepsilon\Gamma_1\Gamma_{45}-4\varepsilon\xi\Gamma_{45}\Gamma_{03})]\varphi(\xi)$$

16. $\langle G_1+P_0,\, P_0+P_3,\, P_1\rangle$

$$\Psi(x)=\exp[\varepsilon m\Gamma_{45}(\Gamma_1x_1+\tfrac{1}{2}\eta\Gamma_{03})]\exp[m\Gamma_2x_2(3\varepsilon\Gamma_{45}+\varepsilon\xi\Gamma_{45}\Gamma_{03}\Gamma_1-1)]\varphi(\xi)$$

17. $\langle G_1+P_0,\, P_1+\alpha P_2,\, P_0+P_3\rangle$

$$\Psi(x)=\exp\{\varepsilon m\Gamma_{45}[\tfrac{1}{2}\eta\Gamma_{03}+(x_2/\alpha)(\Gamma_1+\alpha\Gamma_2)]\}$$
$$\times\exp[\![[m/(\alpha^2+1)](1-2\varepsilon\Gamma_{45})[(\Gamma_2-\alpha\Gamma_1)-\alpha\xi\Gamma_{03}+\varepsilon(\alpha\Gamma_1-\Gamma_2)\Gamma_{45}]$$
$$\times\{\varepsilon\Gamma_{45}[\Gamma_0\Gamma_3+(1/\alpha)\Gamma_1\Gamma_2+\Gamma_{03}\Gamma_1-1]-1\}(\alpha x_1-x_2)]\!]\varphi(\xi)$$

18. $\langle J_{03}+\alpha J_{12},\, P_0,\, P_3\rangle$

$$\Psi(x)=\exp[\varepsilon m\Gamma_{45}(\Gamma_3x_3-\Gamma_0x_0)]\exp[(1/2\alpha)(\Gamma_0\Gamma_3+\alpha\Gamma_1\Gamma_2)$$
$$\times\tan^{-1}(x_1/x_2)]\varphi(x_1^2+x_2^2)$$

19. $\langle J_{03}+\alpha J_{12},\, P_1,\, P_2\rangle$

$$\Psi(x)=\exp[\varepsilon m\Gamma_{45}(\Gamma_1x_1+\Gamma_2x_2)]\exp[-\tfrac{1}{2}(\Gamma_0\Gamma_3+\alpha\Gamma_1\Gamma_2)\ln\xi]\varphi(x_0^2-x_3^2)$$

20. $\langle G_1,\, G_2,\, P_0+P_3\rangle$

$$\Psi(x)=\exp(\tfrac{1}{2}\varepsilon m\eta\Gamma_{45}\Gamma_{03})\exp\{-\tfrac{1}{2}(m/\xi)\Gamma_{03}[\varepsilon(x_1^2+x_2^2)\Gamma_{45}+\Gamma_1x_1+\Gamma_2x_2]\}\varphi(\xi)$$

21. $\langle G_1+P_2,\, G_2+\alpha P_1+\beta P_2,\, P_0+P_3\rangle$

$$\Psi(x)=\exp(\tfrac{1}{2}\varepsilon m\eta\Gamma_{45}\Gamma_{03})[f_1+\varepsilon\Gamma_{45}(g_1\Gamma_1+g_2\Gamma_2+g_3\Gamma_{03})$$
$$+\Gamma_{03}(h_1\Gamma_1+h_2\Gamma_2)+\varepsilon u\Gamma_{45}\Gamma_{03}\Gamma_1\Gamma_2]\varphi(\xi)$$

where

$$f_1=1\qquad g_1=(\alpha m/\tau)(\xi x_2-x_1)\qquad g_2=(m/\tau)[\xi x_1+(\beta\xi-\alpha)x_2]$$
$$h_1=(1/2\tau)[\alpha x_2-(\xi+\rho)x_1]\qquad h_2=(1/2\tau)(x_1-\xi x_2)$$
$$g_3=-(m/2\tau)(\alpha+1)x_1x_2\qquad u=-(m/2\tau)(-x_1^2+\alpha x_2^2+\beta x_1x_2)$$
$$\tau=\xi(\xi+\beta)-\alpha$$

22. $\langle G_1,\, G_2+P_1+\beta P_2,\, P_0+P_3\rangle$

$$\Psi(x)=\exp(\tfrac{1}{2}\varepsilon m\eta\Gamma_{45}\Gamma_{03})\exp[-(x_1/2\xi)\Gamma_{03}(\varepsilon mx_1\Gamma_{45}+\Gamma_1)]$$
$$\times\exp\{[\varepsilon mx_2^2/2(\xi+\beta)]\Gamma_{45}\Gamma_{03}\}\exp[\![[x_2/(\xi+\beta)^2]\{\varepsilon\Gamma_{45}[(1/2\xi)\Gamma_{03}\Gamma_1$$
$$+m(\Gamma_1+\beta\Gamma_2)](\xi+\beta)+[(1-\xi)/2\xi](\xi+\beta-\varepsilon\beta\Gamma_{45})\Gamma_{03}\Gamma_1\}]\!]\varphi(\xi)$$

23. $\langle G_1,\, G_2+P_2,\, P_0+P_3\rangle$

$$\Psi(x)=\exp(\tfrac{1}{2}\varepsilon m\Gamma_{45}\Gamma_{03}\eta)\exp[-(x_1/2\xi)\Gamma_{03}(\varepsilon mx_1\Gamma_{45}+\Gamma_1)]$$
$$\times\exp\{[\varepsilon mx_2^2/2(\xi+1)]\Gamma_{45}\Gamma_{03}]$$
$$\times\exp\{[x_2/(\xi+1)^2][\varepsilon m(\xi+1)\Gamma_{45}\Gamma_2+\tfrac{1}{2}(\varepsilon\Gamma_{45}-\xi-1)\Gamma_{03}\Gamma_2]\}\varphi(\xi)$$

24. $\langle J_{03}, G_1, P_2\rangle$

$$\Psi(x)=\exp(\varepsilon m\Gamma_{45}\Gamma_2 x_2)\exp[-(x_1/2\xi)\Gamma_{03}\Gamma_1]$$
$$\times\exp(-\tfrac{1}{2}\Gamma_0\Gamma_3\ln\xi)\varphi(x_0^2-x_1^2-x_3^2)$$

25. $\langle J_{03}+\alpha P_1+\beta P_2, G_1, P_0+P_3\rangle$

$$\Psi(x)=\exp(\tfrac{1}{2}\varepsilon m\Gamma_{45}\Gamma_{03}\eta)\exp[-(x_1/2\xi)\Gamma_{03}(\varepsilon m x_1\Gamma_{45}+\Gamma_1)]$$
$$\times\exp\{(mx_2/\xi^2)[-\xi\Gamma_2+\beta(1+\Gamma_{45})\Gamma_{03}][\xi\Gamma_{45}(\Gamma_0\Gamma_3-1)-\xi$$
$$+\Gamma_{03}\Gamma_{45}(\alpha\Gamma_1+\beta\Gamma_2)+\Gamma_{03}]\}\varphi(\xi)$$

26. $\langle J_{12}+P_0+P_3, G_1, G_2\rangle$

$$\Psi(x)=\exp\{-(1/2\xi)\Gamma_{03}(\Gamma_1x_1+\Gamma_2x_2)\}$$
$$\times\exp\{[(x_0^2-\boldsymbol{x}^2)/2\xi][\varepsilon m\Gamma_{45}\Gamma_{03}+\tfrac{1}{2}\Gamma_1\Gamma_2(\varepsilon\Gamma_{45}-1)\}\varphi(\xi)$$

27. $\langle J_{03}+\alpha J_{12}, G_1, G_2\rangle$

$$\Psi(x)=\exp[(1/2\xi)(\Gamma_1x_1+\Gamma_2x_2)\Gamma_{03}]$$
$$\times\exp[-\tfrac{1}{2}(\Gamma_0\Gamma_3+\alpha\Gamma_1\Gamma_2)\ln\xi]\varphi(x_0^2-\boldsymbol{x}^2).$$

The following notations were used in the above ansätze:

$$G_\kappa=J_{0\kappa}+J_{\kappa 3}\qquad \kappa=\overline{1,2}$$
$$\xi=x_3+x_0\qquad \eta=x_3-x_0$$
$$\Gamma_{03}=\Gamma_0+\Gamma_3\qquad \Gamma_{45}=\Gamma_4+\Gamma_5$$

α and β are constants, $\varphi(z)$ is a new unknown spinor and $\langle Q_1, Q_2, Q_3\rangle$ is a subalgebra of the algebra (3) and (4) having basis elements Q_1, Q_2, Q_3.

Let us adduce an example of reduced ODE. If one substitutes ansatz 8 into (2) then the equation for $\varphi(z)$ becomes

$$[2z^{1/2}\Gamma_2\,\mathrm{d}/\mathrm{d}z+\tfrac{1}{2}z^{-1/2}\Gamma_2-m+2\varepsilon m(\Gamma_4+\Gamma_5)]\varphi(z)=0.$$

Note 1. If one puts $\varepsilon=0$ in (3) then (3) and (4) generate the local Lie group P(1, 3). That is why, on putting $\varepsilon=0$ into the ansätze above, one obtains Poincaré-invariant ansätze for the spinor field constructed in [3].

Note 2. The above non-local ansätze can be applied to the construction of exact solutions of non-linear Lorentz-invariant spinor equations admitting the group (5). One example of such equations is

$$\{\partial_\mu\partial^\mu+\lambda[\bar\Psi(\Gamma_4+\Gamma_5)\Gamma_\mu\partial_\mu\Psi](\Gamma_4+\Gamma_5)\}\Psi=0$$

where λ is constant and $\bar\Psi=\Psi^{\mathrm{T}}\Gamma_0\Gamma_4$. This problem will be considered in a future publication.

References

[1] Fushchich W I and Nikitin A G 1987 *Symmetries of Maxwell's Equations* (Dordrecht: Reidel)
[2] Fushchich W I and Zhdanov R Z 1987 *J. Phys. A: Math. Gen.* **20** 4173
[3] Fushchich W I and Zhdanov R Z 1988 *Fiz. Elem. Cast. Atom. Jadra* (USSR) **19** to appear
[4] Patera J, Winternitz P and Zassenhaus H 1975 *J. Math. Phys.* **16** 1597
[5] Kamran N, Légaré M, McLenaghan R G and Winternitz P 1988 *J. Math. Phys.* **29** 403
[6] Fushchich W I 1971 *Teor. Mat. Fiz.* **7** 3
[7] Fushchich W I 1974 *Lett. Nuovo Cimento* **11** 508
[8] Fushchich W I and Zhdanov R Z 1987 *Symmetry and Solutions of Nonlinear Equations of Mathematical Physics* (Kiev: Mathematical Institute) p 17